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THEORY OF THE INVESTIGATION OF
THE ATMOSPHERE OF BINARY STARS
BY ANALOGY BETWEEN GAS DYNAMICS
AND SHALLOW-WATER FLOWS

by Ernst W. Adams and Burkhard Schulz-Jander

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BETWEEN GAS DYNAMICS AND SHALLOW-WATER FLOWS

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ABSTRACT

Approximate solutions of the differential equations of continuum isentropic gas dynamics may be obtained for flows in the equatorial plane of a binary star atmosphere, by analogy between gas dynamics and shallow-water flows in a rotating water tank. The local inclination of the tank's bottom is proportional to the resulting local gravitational force at the corresponding point in the binary-star atmosphere. The analogy requires a constant polytropic coefficient of two. This application of the analogy rests on its extension to rotating systems with external force fields. A theoretical evaluation of the analogy (for the earth's atmosphere under hydrostatic conditions with polytropic coefficient equal to two) yields the correct "maximum" height of the atmosphere—where density reaches zero. This study is motivated by the possibility that it may be applied to the early stages of the evolution of the earth-moon system after its hypothetical fission.

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THEORY OF THE INVESTIGATION OF THE ATMOSPHERE OF BINARY STARS BY ANALOGY BETWEEN GAS DYNAMICS AND SHALLOW-WATER FLOWS

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INTRODUCTION

According to observation, a binary star system may possess a common atmosphere which envelops both stars. The motion of a particle in such an atmosphere may be treated by the restricted three-body problem in celestial mechanics. Several publications have used this approach, e.g., Kopal (1956, 1957) and Mrs. Gould (1957, 1959). According to Prendergast (1960), the mean free path in these atmospheres is of the order of magnitude of 1 to 10 km; i.e., this characteristic length is so much smaller than either the separation of the stars or the radius of one of them that collisions between particles in the atmosphere must be taken into account. Therefore, Prendergast (1960) studied stationary solutions of the differential equations for continuum isentropic gas flows in the equatorial plane of the binary system, by neglecting the pressure gradient and the velocity component normal to the "Lagrangian surfaces." Subsequently, Huang (1965) employed a steady-state celestial-mechanics approach, which neglects the pressure gradient but takes into account collisions between particles—using the statistical properties of the Jacobian constants of the colliding particles.

The theory presented in this paper was developed because of a desire to study gaseous motions in the earth-moon vicinity during the early stages after a fission of their hypothetical parent body. By such motions, mass, energy, and angular momentum would be redistributed and possibly exported from this binary system.

The relatively large mean free path quoted above may be assumed to represent an average value for the entire atmosphere enveloping the stars revolving around each other. It therefore seems desirable to give a rigorous hydrodynamic treatment of the atmosphere, including the pressure gradient, at least for the regions of maximum density. In addition, the observed eruption phenomena in binary systems, e.g., Huang (1965), render a nonstationary treatment of the atmosphere desirable. If these atmospheric motions are always symmetrical with respect to the equatorial

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plane, the differential equations of motion and energy for three-dimensional isentropic gas flows reduce rigorously to their two-dimensional versions in this plane; in the continuity equation, however, one term must be neglected in this two-dimensional transient approach. Prendergast (1960) points out that the total mass of the atmosphere is concentrated about the equatorial plane; therefore, limiting the study to flows in this plane appears to be a relatively unimportant restriction of generality which, also, affects the publications mentioned above. Even in the equatorial plane, the time-dependent nonlinear system of differential equations with three independent variables precludes any analytic solutions. Numerical solutions reveal general characteristics only after very many individual cases have been computed and evaluated. Because of this situation, the analogy between gas dynamics and shallow-water motions is employed here to furnish the theoretical background for a water-tank experiment simulating transient flows in the equatorial plane of the binary-star atmosphere.* This requires a discussion of shallow-water flows and an extension of this analogy to flows in rotating systems subjected to external force fields. An analogy such as the one presented here should be illustrated by some typical and sufficiently general results. Unfortunately means are not available to carry out these experiments for the time being. Since this situation will not change in the foreseeable future, it was decided to publish the paper now. As a restricted test of validity and accuracy of the method presented, the hydrostatic density distribution in the earth's atmosphere is obtained in a special application as discussed further on.

SHALLOW-WATER FLOWS

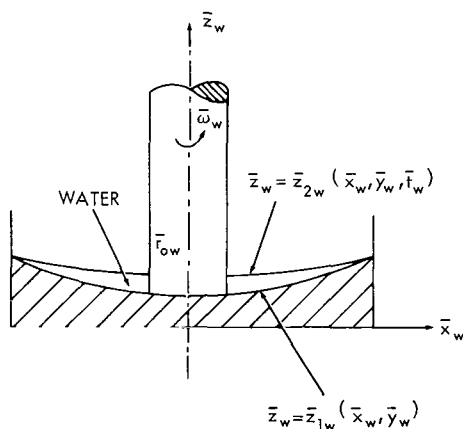


Figure 1—Water tank.

A container (Figure 1) is filled to a certain height with water. At time $\bar{t}_w = 0$, this water tank is rotating about the \bar{z}_w -axis with constant angular velocity $\bar{\omega}_w$. The acceleration of gravity \bar{g}_w is assumed to act in the negative \bar{z}_w -direction. A cylinder with radius \bar{r}_{0w} is mounted at the bottom of the container; its axis coincides with the \bar{z}_w -axis. Along the wetted surfaces of the cylinder and/or the bottom, sources or sinks may add water to the system or drain it off according to a law that may vary locally and in time. (The total amount of water in the container need not necessarily be constant.) The shape of the bottom will be treated later.

The subscripts w and g denote water and gas, respectively; the subscript i stands for w as well as g . Dimensional properties are marked by an upper bar. The angular velocities $\bar{\omega}_i$, the lengths \bar{r}_{0i} , and the densities $\bar{\rho}_{0i}$ will be used as constant reference quantities. Non-dimensional Cartesian coordinates x_i, y_i, z_i , velocities u_i, v_i, w_i , times t_i , pressures p_i , and densities ρ_i are introduced by relations

*For general information about this analogy, Courant/Friedrichs (1948) and Wehausen/Laitone (1960) are recommended.

such as

$$\begin{aligned} x_i &= \frac{\bar{x}_i}{\bar{r}_{0i}}, & u_i &= \frac{\bar{u}_i}{\bar{r}_{0i} \bar{\omega}_i}, & t_i &= \bar{\omega}_i \bar{t}_i, & p_i &= \frac{\bar{p}_i}{\bar{\rho}_{0i} \bar{\omega}_i^2 \bar{r}_{0i}^2}, \\ \rho_i &= \frac{\bar{\rho}_i}{\bar{\rho}_{0i}} \text{ with } \bar{\rho}_w = \bar{\rho}_{0w} = \text{const. and } \bar{\rho}_{0g} = \bar{\rho}_g(\bar{r}_{0g}). \end{aligned} \quad (1)$$

A Froude-number \mathfrak{F}_i is introduced as the ratio of the centrifugal acceleration $\bar{r}_{0i} \bar{\omega}_i^2$, at distance \bar{r}_{0i} from the axis of rotation, to the gravitational acceleration \bar{g}_i :

$$\mathfrak{F}_i = \bar{r}_{0i} \bar{\omega}_i^2 / \bar{g}_i = \text{const.}, \quad (2)$$

The symbol D/Dt_w denotes the substantial derivative with respect to time t_w . According to Lamb (1932, p. 4 and p. 318), the equations of motion of the water flow in a system rotating at constant angular velocity $\bar{\omega}_w$ read as follows:

$$\begin{aligned} \frac{Du_w}{Dt_w} - 2v_w - x_w &= - \frac{\partial p_w}{\partial x_w}, \\ \frac{Dv_w}{Dt_w} + 2u_w - y_w &= - \frac{\partial p_w}{\partial y_w}, \end{aligned} \quad (3)$$

and

$$\frac{Dw_w}{Dt_w} = - \frac{\partial p_w}{\partial z_w} - \frac{1}{\mathfrak{F}_w}. \quad (4)$$

Here Equation 4 was obtained from

$$\bar{\rho}_w \frac{D\bar{w}_w}{D\bar{t}_w} = - \frac{\partial \bar{p}_w}{\partial \bar{z}_w} - \bar{\rho}_w \bar{g}_w, \quad (5)$$

using Equations 1. The left-hand sides of Equations 3 through 5 express the mass-fixed accelerations in a system at rest, provided that the quantities of the rotating system are used. According to Goldstein (1960, p. 12), the equation of continuity is independent of the chosen coordinate system, since it simply states the constancy of a scalar quantity. It thus reads

$$\frac{\partial u_w}{\partial x_w} + \frac{\partial v_w}{\partial y_w} + \frac{\partial w_w}{\partial z_w} = 0. \quad (6)$$

For $\bar{\omega}_w = 0$, the basic simplifying assumptions of the shallow-water theory are (see Wehausen and Laitone, 1960, p. 667):

- I. Irrespective of flows in the water, the pressure distribution in a vertical column of water is the same as in hydrostatics, and
- II. The horizontal velocity components u_w and v_w are independent of z_w . (Considering the no-slip condition at the bottom, this assumption can be satisfied only approximately and for short times in every version of the analogy.)

In addition to assuming the validity of I and II for $\bar{\omega}_w \neq 0$, the following assumption must be made:

- III. The hydrostatic pressure distribution holds true even in the presence of motions relative to the rotating frame of reference.

If the fluid rotates as a solid body, its free surface (subscript II) is given by

$$\bar{z}_{II}(\bar{r}_w) = \text{const.} + \bar{r}_w^2 \bar{\omega}_w^2 / 2\bar{g}_w ,$$

where

$$\bar{r}_w^2 = \bar{x}_w^2 + \bar{y}_w^2 , \quad (7)$$

If any flows relative to the rotating frame of reference ("disturbances") are superimposed to this solid body rotation, the free surface is given by

$$z_w = z_{2w}(x_w, y_w, t_w) = \eta_w(x_w, y_w, t_w) + z_{II}(x_w, y_w) .$$

If the vertical accelerations Dw_w/Dt_w caused by these disturbances are negligible because of a sufficiently small water depth, then Equation 5 is satisfied by the hydrostatic distribution specified in assumption III,

$$\bar{p}_w = \bar{\rho}_w \bar{g}_w \left[\bar{z}_{II}(\bar{x}_w, \bar{y}_w) + \bar{\eta}_w(\bar{x}_w, \bar{y}_w, \bar{t}_w) - \bar{z}_w \right] + \bar{p}_{00} , \quad (8)$$

where \bar{p}_{00} is the atmospheric pressure acting upon the free surface. Equation 8 has been derived by Lamb (1932, p. 318) for rotating shallow water flows. A nondimensional form of Equation 8 is given by

$$p_w = p_{00} + \left[(z_{2w} - z_{1w}) + (z_{1w} - z_w) \right] / \bar{g}_w . \quad (9)$$

The functions $F_{1w} = z_w - z_{1w}(x_w, y_w) = 0$ and $F_{2w} = z_w - z_{2w}(x_w, y_w, t_w) = 0$ represent the wetted bottom and the free water surface, respectively (see Figure 1). Therefore,

$$DF_{jw}/Dt_w = 0, \quad j = 1, 2, \quad (10)$$

according to Lamb (1932, p. 7), since these surfaces are always composed of the same fluid particles, and thus

$$w_{jw} = \frac{\partial z_{jw}}{\partial t_w} + u_{jw} \frac{\partial z_{jw}}{\partial x_w} + v_{jw} \frac{\partial z_{jw}}{\partial y_w}, \quad j = 1, 2. \quad (11)$$

Because of assumption II and Equation 11, integration of the continuity Equation 6 with respect to z_w in the limits z_{1w} and z_{2w} gives

$$-\frac{\partial(z_{2w} - z_{1w})}{\partial t_w} + \frac{\partial}{\partial x_w} [(z_{2w} - z_{1w})u_w] + \frac{\partial}{\partial y_w} [(z_{2w} - z_{1w})v_w] = 0. \quad (12)$$

For $\bar{\omega}_w = 0$, Equations 3 and 12 represent the shallow-water theory, which is a first-order approximation, by expanding in powers of a parameter ϵ_1 , defined as the ratio of the squares of the characteristic horizontal and vertical scales of length (Wehausen and Laitone, 1960, p. 466 and 667). Subsequently, the wavelength $\bar{\lambda}_w$ and the amplitude $\bar{\eta}_w$ of the disturbance are employed for these scales. According to Laitone (1962), a second-order approximation yields in case of $\bar{\lambda}_w \rightarrow 0$ and $\bar{\omega}_w = 0$

$$\frac{\partial \bar{p}_w}{\partial \bar{z}_w} \propto -\bar{\rho}_w \bar{g}_w \left[1 - \frac{3}{2} \left(\frac{\bar{\eta}_w}{\bar{H}_w} \right)^2 \right] \quad \text{and} \quad \bar{H}_w \frac{\partial \bar{u}_w}{\partial \bar{z}_w} \propto \sqrt{\bar{g}_w \bar{H}_w} \left(\frac{\bar{\eta}_w}{\bar{H}_w} \right)^2,$$

where

$$\bar{H}_w = \bar{z}_{II} - \bar{z}_{Iw}. \quad (13)$$

Correspondingly, Laitone (1962) shows for $\bar{\lambda}_w \rightarrow 0$ and $\bar{\omega}_w = 0$

$$\frac{\partial \bar{p}_w}{\partial \bar{z}_w} \propto -\bar{\rho}_w \bar{g}_w \left[1 - \left(\frac{2\pi \bar{H}_w}{\bar{\lambda}_w} \right)^2 \frac{\bar{\eta}_w}{\bar{H}_w} \right] \quad \text{and} \quad \bar{H}_w \frac{\partial \bar{u}_w}{\partial \bar{z}_w} \propto \sqrt{\bar{g}_w \bar{H}_w} \left(\frac{2\pi \bar{H}_w}{\bar{\lambda}_w} \right)^2 \frac{\bar{\eta}_w}{\bar{H}_w}. \quad (14)$$

Equations 13 and 14 show for $\bar{\omega}_w = 0$ that the following quantities have to be sufficiently small in order to allow an inviscid water flow to be approximated by the shallow-water theory:

- the hydrostatic depth $\bar{H}_w(\bar{x}_w, \bar{y}_w) = \bar{z}_{II} - \bar{z}_{Iw}$ of the water,
- the ratio $\bar{\eta}_w/\bar{H}_w$ of the disturbance amplitude $\bar{\eta}_w$ to \bar{H}_w , and
- the ratio of $\bar{H}_w/\bar{\lambda}_w$.

It is assumed here that conditions a, b, and c also govern the validity of the shallow-water theory in the rotating water tank, provided that both $\bar{\omega}_w$ and $|\text{grad } z_{Iw}(\bar{x}_w, \bar{y}_w)|$ are sufficiently small.

THE ANALOGY BETWEEN SHALLOW-WATER FLOWS AND GAS FLOWS

By assuming $z_g = w_g = 0$, a two-dimensional transient gas flow (subscript g) will be related to the three-dimensional transient shallow-water flow (subscript w). By use of $H_w(1)$ as defined in Equation 13 and taken at station $x_w^2 + y_w^2 = 1$, the following functional relationships now are introduced between the water flows and gas flows:

$$x_g = x_w, \quad y_g = y_w, \quad u_g = u_w, \quad v_g = v_w, \quad t_g = t_w, \quad (15)$$

$$\rho_g = \left[z_{2w}(x_w, y_w, t_w) - z_{1w}(x_w, y_w) \right] / H_w(1), \quad (16)$$

and

$$p_g = \left[z_{2w}(x_w, y_w, t_w) - z_{1w}(x_w, y_w) \right]^2 / 2\beta_w H_w(1) = \frac{\rho_g^2 H_w(1)}{2\beta_w}. \quad (17)$$

The scales of length and time of the two flows under consideration are connected by the quotients $\bar{r}_{0w}/\bar{r}_{0g}$ and $\bar{t}_w/\bar{t}_g = \bar{\omega}_g/\bar{\omega}_w = \text{const.}$, respectively. Since the isentropic flow of a chemically homogeneous gas is considered here, there exists a functional relation $p_g = p_g(\rho_g)$ between pressure and density. The special polytropic relation $p_g \propto \rho_g^\kappa$ with $\kappa = 2$, which follows immediately from Equations 16 and 17, can only hold true approximately since the kinetic theory of gases shows that $1 < \kappa \leq 1.67$.

Substitution of Equations 15, 16, and 17 in Equations 3 and 12 yields, by use of Equation 9,

$$\rho_g \left(\frac{Du_g}{Dt_g} - 2v_g - x_g \right) = - \frac{\partial p_g}{\partial x_g} - \rho_g \frac{\partial (z_{1w}/\beta_w)}{\partial x_g}, \quad (18)$$

$$\rho_g \left(\frac{Dv_g}{Dt_g} + 2u_g - y_g \right) = - \frac{\partial p_g}{\partial y_g} - \rho_g \frac{\partial (z_{1w}/\beta_w)}{\partial y_g}, \quad (19)$$

and

$$\frac{\partial \rho_g}{\partial t_g} + \frac{\partial}{\partial x_g} (\rho_g u_g) + \frac{\partial}{\partial y_g} (\rho_g v_g) = 0. \quad (20)$$

The gas flow defined by Equations 15, 16, and 17 is two-dimensional, since the third coordinate z_w of the corresponding three-dimensional water flow represents the density ρ_g of the gas. Because

$$\frac{\partial w_w}{\partial z_w} \neq 0,$$

it does not follow from Equations 6 and 20 that ρ_g is constant, even though this seems to be suggested by Equations 1 and 15. By the argument following Equation 5, the left-hand sides of Equations 18 and 19 are the mass-fixed accelerations in the rotating system of reference, and Equation 20 is the two-dimensional equation of continuity of a compressible medium in this system. The last terms in the right-hand sides of Equations 18 and 19 may be interpreted as due to a conservative body force, which can be derived from a potential $z_{1w}(x_g, y_g)/\beta_w$. Thus Equations 18, 19 and 20 are the equations of motion and continuity (in the continuum regime) of two-dimensional gas flows subjected to conservative body forces in a system rotating at a constant angular velocity.

The first law of thermodynamics yields for any mass element of the supposedly isentropic gas flow the relation $\bar{c}_{pg} d\bar{T}_g = d\bar{p}_g/\bar{\rho}_g$ between the specific heat \bar{c}_{pg} at constant pressure, the absolute temperature \bar{T}_g , the pressure \bar{p}_g , and the density $\bar{\rho}_g$ of the gas. Therefore, the energy equation in the rotating system, because of the argument following Equation 5, reads

$$\bar{\rho}_g \bar{c}_{pg} \frac{D\bar{T}_g}{Dt_g} = \frac{D\bar{p}_g}{Dt_g} \quad \text{if} \quad c_{pg} = \text{const.} \quad (21)$$

Equation 21 is satisfied by two-dimensional, transient calorically perfect gas flows considered here. Thus the relation between water and gas flows as given by Equations 15, 16 and 17 is a complete analogy within the limitations of the assumptions mentioned earlier.

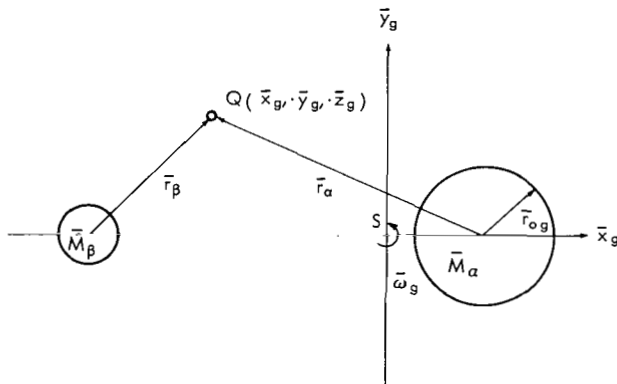
The generalization of the analogy developed earlier consists of admitting (a) conservative body forces, which determine the bottom shape $z_w = z_{1w}(x_w, y_w)$ of the water tank, and (b) rotating systems. Variable bottom configurations have been used in many applications of the analogy to problems without body forces in order to admit arbitrary constant values of κ . From 1952 to 1962, this issue initiated a discussion in which body forces were not mentioned at all. In agreement with the derivation presented earlier, Laitone (1952, 1953, 1961, 1962) and Wehausen and Laitone (1960) pointed out the necessity of $z_{1w}(x_w, y_w) = \text{const.}$ in cases without body forces which were the only ones they considered in this context. Loh (1959, 1962) and Bryant (1960, 1962) remarked that $z_{1w}(x_w, y_w) \neq \text{const.}$ is admissible in view of the other approximations involved in the analogy, such as constant values of κ .

The conditions for the validity of the shallow-water theory mentioned earlier also apply to the analogy. Experiments by Laitone (1952) confirmed that the analogy yields satisfactory agreement between gas dynamics and water flows, for tap water with usual surface tension, as long as the water depth is less than 1/4 inch and the wavelength $\bar{\lambda}_w$ of the disturbances exceeds 3 inches; see also Laitone (1953, 1961, 1962). Under consideration of the surface tension of water, Gupta's theory (1965) yields about the same limits for the validity of the analogy carried out in nonrotating water tanks.

For sufficiently small values of $\bar{\omega}_w$, the wave systems caused by the tank's rotation (e.g., Lamb, 1932) presumably are negligible as compared with the wave systems discussed earlier for $\bar{\omega}_w = 0$. Since, in addition, a transient boundary layer growth at the tank bottom is caused by the waves disturbing the solid-body rotation of the water, the relation $\bar{t}_g \bar{\omega}_g = \bar{t}_w \bar{\omega}_w$ following from Equations 1 and 15 prohibits atmospheric motions to be studied in the water tank beyond sufficiently short intervals \bar{t}_g after the beginning of these waves. The required small values of $\bar{\omega}_w$ demand a sufficiently small maximum inclination of the tank bottom. This may necessitate the addition of a detergent to the water because of the surface tension.

APPLICATION TO AN ASTROPHYSICAL AND A GEOPHYSICAL PROBLEM

The right-hand sides of Equations 18 and 19 were interpreted as representing body forces which result from a potential $z_{1w}(x_g, y_g)/\bar{\omega}_w$. The gravitational potential of spherically symmetric celestial bodies will now be employed to demonstrate the analogy. Figure 2 shows a binary system of two celestial bodies rotating about their common center of gravity S at the constant angular velocity $\bar{\omega}_g$. The masses of the bodies are \bar{M}_α and \bar{M}_β ; \bar{r}_α and \bar{r}_β denote the distance of any point Q($\bar{x}, \bar{y}, \bar{z}$) from the centers of the two masses, respectively. The gravitational potential in the space outside \bar{M}_α and \bar{M}_β then is given by



$$\bar{\Psi} = \frac{\bar{\Gamma} \bar{M}_\alpha}{\bar{r}_\alpha} + \frac{\bar{\Gamma} \bar{M}_\beta}{\bar{r}_\beta}, \quad (22)$$

Figure 2—Configuration of celestial bodies.

where $\bar{\Gamma}$ is the universal gravitational constant.

In the following, the contribution of atmospheric matter to the gravitational potentials will be neglected. The mass ratio $\mu = \bar{M}_\beta/\bar{M}_\alpha$ is introduced, and the radius of any one of the two celestial bodies is used as a constant reference length \bar{r}_{0g} ; i.e., $r_\alpha = \bar{r}_\alpha/\bar{r}_{0g}$ and $r_\beta = \bar{r}_\beta/\bar{r}_{0g}$. Since $\bar{\Psi}$ has the units ($\text{cm}^2 \text{sec}^{-2}$) the expression $\bar{r}_{0g}^2 \bar{\omega}_g^2 = \text{const.}$ is employed to obtain the nondimensional potential

$$\Psi = \frac{1}{\bar{\omega}_g^2} \left(\frac{1}{r_\alpha} + \frac{\mu}{r_\beta} \right), \quad (23)$$

where the Froude number

$$\mathfrak{F}_\alpha = \frac{\bar{r}_{0g}^3 \bar{\omega}_g^2}{\bar{\Gamma} \bar{M}_\alpha}$$

is the ratio of centrifugal to gravitational forces at the surface of the celestial body \bar{M}_α . In applications of the analogy, $\mathfrak{F}_\alpha = \mathfrak{F}_g$. In the equatorial plane, the components of the gradient of this potential will now be equated to the last terms in Equations 18 and 19 to obtain an expression for the shape of the bottom of the water tank:

$$-\text{grad } \frac{z_{1w}}{\mathfrak{F}_w} = \text{grad } \Psi. \quad (24)$$

With z_0 as a constant of integration, it is found that

$$z_0 - z_{1w} = \frac{\mathfrak{F}_w}{\mathfrak{F}_\alpha} \left(\frac{1}{r_\alpha} + \frac{\mu}{r_\beta} \right). \quad (25)$$

The analogy as expressed by Equations 15, 16, and 17 enables one to study in the equatorial plane two-dimensional atmospheric motions subjected to a gravitational field $\bar{\Psi}(\bar{x}_g, \bar{y}_g, \bar{z}_g)$ by experimental simulation in a water tank with the bottom $z_w = z_{1w}(x_w, y_w)$. These atmospheric flows are restricted to the equatorial plane $z_g = 0$, where the complete differential equations (18), (19), and (21) take on special two-dimensional forms, provided that the motion is isentropic and symmetrical with respect to the plane $z_g = 0$. Because of this symmetry, the first derivatives with respect to z_g of u_g, v_g, p_g, ρ_g , and T_g vanish identically as functions of x_g, y_g , and t_g in the plane $z_g = 0$. Since $\partial w_g / \partial z_g \neq 0$ for $z_g = 0$, the three-dimensional equation of continuity retains this term $\partial w_g / \partial z_g$, which is absent in Equation 20. If

$$\left[\left(\partial w_g / \partial z_g \right) \right]_{z_g=0}$$

is known as a function of x_g, y_g , and t_g by use of some information outside the analogy, this term can be represented by a suitable distribution of sources and sinks in the bottom $z_w = z_{1w}(x_w, y_w)$ of the water tank. The equation of continuity (Equation 20) then possesses a corresponding additional source term.

At present, the authors cannot experimentally test the accuracy of the analogy; a theoretical check is therefore desirable. Such a test of accuracy is provided by the application of the analogy to the earth's atmosphere under conditions of a known density distribution; i.e., in case of the hydrostatic atmosphere. In this case, the foregoing equations simplify somewhat, since $\mu = 0$ and $\bar{\Gamma} \bar{M}_\alpha / \bar{r}_{0g}^2 = \bar{g}_0$, where \bar{g}_0 is the gravitational acceleration at the surface of the earth. Then the

quotient $\mathfrak{F}_w/\mathfrak{F}_a$ reduces to an expression defining the constant A:

$$\frac{\mathfrak{F}_w}{\mathfrak{F}_a} = \frac{\bar{r}_{0w} \bar{\omega}_w^2}{\bar{r}_{0g} \bar{\omega}_g^2} = A, \quad (26)$$

i.e., the ratio of the centrifugal forces in the container to those at the earth's equator. Because of Equation 15, $r_a = r_g = r_w$. Thus, by taking $z_{1w} = 0$ at $r_w = 1$, Equation 25 yields

$$z_{1w} = A \left(1 - \frac{1}{r_w} \right). \quad (27)$$

The atmosphere enveloping the earth is assumed to rotate as a solid body, such that in the analogy $z_{2w} = z_{11w}$ and $\eta_w = 0$. From Equation 7 it is thus found

$$z_{11w} = \text{const.} + \frac{\mathfrak{F}_w}{2} r_w^2. \quad (28)$$

Because of Equations 16 and 17,

$$\frac{p_g}{\rho_g} = \frac{1}{2\mathfrak{F}_w} (z_{11w} - z_{1w}). \quad (29)$$

The equation of state for a thermally perfect gas, $\bar{p}_{0g} = \bar{\rho}_{0g} \bar{R}_g \bar{T}_{0g}$, requires at the surface of the earth (subscript 0)

$$\frac{p_{0g}}{\rho_{0g}} = \frac{\bar{R}_g \bar{T}_{0g}}{\bar{g}_0 \bar{r}_{0g}} \frac{1}{C_g}. \quad (30)$$

Because $\mathfrak{F}_a = \mathfrak{F}_g$, and $z_{1w} = 0$ at $r_w = 1$, Equations 29 and 30 yield

$$z_{11w} \Big|_{r_w=1} = 2AB, \quad (31)$$

with a constant B defined by

$$B = \frac{\bar{R}_g \bar{T}_{0g}}{\bar{g}_0 \bar{r}_{0g}}. \quad (32)$$

From Equations 27, 28, and 31 finally,

$$z_{11w} - z_{1w} = A \left[2B + \frac{\mathfrak{F}_g}{2} (r_w^2 - 1) - \left(1 - \frac{1}{r_w} \right) \right] \quad (33)$$

is obtained to express the nondimensional density distribution in the earth's atmosphere under hydrostatic conditions and for $\kappa = 2$. From left to right, the terms between the brackets in Equation 33 represent the contributions of the earth's surface temperature, the centrifugal force due to the assumed rigid-body rotation, and the gravitational force.

Because of the small ratio of the maximum extension of the atmosphere to the earth's radius, it is reasonable to write $r_g = r_w = 1 + \epsilon$, where second and higher powers of ϵ are to be neglected. From $1/(1 + \epsilon) = 1 - \epsilon + \epsilon^2 \dots$, the following relation is obtained to replace Equation 33:

$$z_{IIw} - z_{Iw} = A \left[2B - (1 - \beta_g) \epsilon \right] . \quad (34)$$

Because of Equation 17, the pressure variation in the earth's atmosphere according to the analogy is

$$\frac{p_g}{p_{0g}} = \left[\frac{z_{IIw} - z_{Iw}}{(z_{IIw} - z_{Iw})_0} \right]^2 = \left[1 - \frac{1 - \beta_g}{2B} \epsilon \right]^2 , \quad (35)$$

where the effect of the centrifugal force is negligible even at the earth's equator since $\beta_g = 3.32 \times 10^{-3}$. Because $B = 1.27 \times 10^{-3}$ for a surface temperature $\bar{T}_{0g} = 293$ ($^{\circ}\text{K}$), the analogy predicts that the atmospheric pressure \bar{p}_g vanishes at a maximum altitude as given by

$$\bar{H}_{\max} = \bar{r}_{0g} \cdot \epsilon_{\max} = \bar{r}_{0g} \cdot \frac{2B}{1 - \beta_g} \approx \frac{1}{394} \bar{r}_{0g} \approx 16 \text{ km} .$$

It will now be shown that this result for \bar{H}_{\max} agrees with the one of the well-known hydrostatic polytropic atmosphere with $\kappa = 2$, which is defined (a) by neglecting rotation (i.e., $\beta_g = 0$) and (b) by assuming a constant acceleration of gravity. Under these conditions, Prandtl/Tietjens (1957, p. 33) show that

$$\frac{p_g}{p_{0g}} = \left(1 - \frac{\kappa - 1}{\kappa} \frac{\bar{h}}{\bar{h}_0} \right)^{\kappa / \kappa - 1} , \quad (36)$$

where, in the notation of this paper,

$$\bar{h} = \bar{r}_g - \bar{r}_{0g} , \quad \text{and} \quad \bar{h}_0 = B \bar{r}_{0g} . \quad (37)$$

Because $\epsilon = \bar{h} / \bar{r}_{0g}$, Equation 26 yields, in the case $\kappa = 2$, the relationship

$$\frac{p_g}{p_{0g}} = \left(1 - \frac{\epsilon}{2B} \right)^2 , \quad (38)$$

which is identical to Equation 35 for $\mathfrak{U}_g = 0$. It should be recalled here that the polytropic hydrostatic atmospheres defined by conditions (a) and (b) mentioned earlier possess a finite maximum altitude \bar{H}_{\max} , with the exception of $\kappa = 1$.

A BRIEF REVIEW OF RESULTS ON MOTIONS IN THE EQUATORIAL PLANE OF A BINARY STAR SYSTEM

Prominence activity in the sun is a well-known phenomenon. Observation of binary stars suggests that the prominence activity of the secondary star is stronger on the hemisphere facing the primary star. Appropriate initial velocity vectors of these prominences could result in a gaseous stream toward the primary or in an escape from the binary system (Sahade, 1960).

Within the context of the restricted three-body problem of celestial mechanics, several authors have studied trajectories of ejected particles in the equatorial plane. For a particle moving in this plane, the Jacobi integral shows the existence of curves of zero velocity in a frame rotating with the binary system. These closed curves are the "Roche equipotentials" on which the geopotential is constant, i.e., the sum of the gravitational potentials of the component stars and the potential of the centrifugal force caused by the rotation of the system. These Roche equipotentials define admissible cross-sections of each component star in the equatorial plane. The equipotential enclosing the largest area in this plane and only one component star is called the Roche limit. For a particle having a given value of the Jacobi integral, the corresponding Roche equipotential represents a barrier which the particle, if moving inside its folds, can never penetrate. The trajectories of particles ejected from a component star completely filling its Roche limit take place in an area of the equatorial plane whose Roche equipotential boundary envelops both component stars. For this case, trajectories of particles have been computed by numerical integrations of the equations of motion of the restricted three-body problem (Kopal, 1959). The results could be summarized as follows: (1) particles fall upon the companion star; (2) particles fall back upon the parent star; and (3) particles escape from the system if they possess sufficiently high initial velocities, generally of the order of several hundred kilometers per second.

According to Sahade (1960), velocities of 700 km/sec have been observed in the binary system V 444 Cygni. Because of the smaller masses, considerably smaller velocities would be sufficient for escape of matter from the earth-moon system after fission of its hypothetical parent body.

These results of celestial mechanics previously referred to do not account for interactions between particles. Therefore, Struve (1957), has pointed out that it would be important to consider the problem at hand also from a hydrodynamical point of view and to allow for the effect of radiation pressure and magnetic forces. The theory presented earlier in this paper furnishes a background for a water tank experiment that attempts to simulate gaseous streams in the equatorial plane of a binary star system by accounting for gravitational, inertial, and hydrodynamic pressure forces. These results would pertain to the continuum flow regime, which is separated from motions caused by applications of celestial mechanics by the rarefied-flow region. The transient water tank experiment, therefore, would furnish a necessary supplement to published results from (a) applications

of celestial mechanics (Kopal, 1959), and (b) stationary continuum flow theory under rather restrictive assumptions (Prendergast, 1960).

COMMENTS WITH RESPECT TO THE EXPERIMENTAL SIMULATION OF THE FLOW IN THE EQUATORIAL PLANE OF THE BINARY STAR SYSTEM

In the water tank, the primary and the secondary stars are represented by cylinders mounted parallel to the table's z -axis of rotation (Figure 3). Since the primary star, by definition, possesses the larger mass, the bottom of the tank reaches its smallest z -coordinate at its intersection with the "primary" cylinder. If the surfaces of the cylinders are sufficiently smooth, the motion of the water is affected very little by rotations of the cylinders about their own axes of symmetry. Therefore, the wetted portions of these surfaces may have to be roughened artificially if this eigen-rotation of the cylinders is to influence the water flow appreciably. Even though the gravitational fields extend to infinity in the equatorial plane, it is sufficient to place a board parallel to the axis of rotation of the system on the bottom of this tank, provided that the board is located beyond the synchronous satellite orbit of the system. A water sink may be placed at the intersection of the board and the bottom.

Under hydrostatic conditions, the water depth at the circumference of one of these cylinders may be prescribed arbitrarily. Because of Equations 32 and 33, this depth is proportional to the temperature of the atmosphere at the surface of the corresponding component star. This temperature should be sufficiently low to ensure the gaseous state of the atmosphere since Equations 16, 17, and the equation of state of a thermally perfect gas show that the temperature of the atmosphere is proportional to its density in the analogy employed here. The tank's bottom between the cylinders does not have to be covered by water continuously in time. As far as mass transfer from the secondary to the primary star is concerned, it is sufficient to have intermittent water jets emanate from the "secondary" cylinder. The mass flow from the secondary to the primary cylinder can be visualized by adding dye to the water pouring out of the secondary cylinder.

CONCLUDING REMARKS

The objective of conducting a water tank experiment should be a qualitative simulation of transient astrophysical effects such as those mentioned earlier in this paper. The following details militate against quantitative validity of the results:

a. the assumption of a constant polytropic coefficient (of 2) which is essential to the analogy in every possible version;

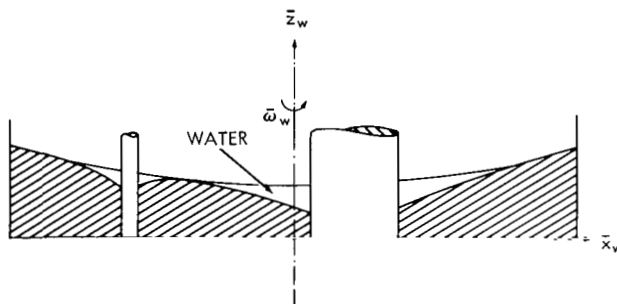


Figure 3—Schematic configuration of water tank suggested to study the atmosphere around the celestial bodies shown in Figure 2.

- b. the assumption of isentropic gas flows;
- c. the treatment of flows in the equatorial plane without regard for the interaction with the general three-dimensional flow field outside of this plane; this shortcoming manifests itself in neglecting the term $\partial w / \partial z$ in the three-dimensional continuity equation;
- d. the omission of radiation pressure, turbulence, viscous friction, heat conduction, diffusion, and other real gas effects;
- e. experimental difficulties, e.g., the ones discussed earlier.

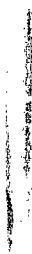
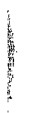
Even though this seems to be a formidable accumulation of shortcomings of the method presented in this paper, the approach may be expected to yield more reliable and realistic results about sufficiently dense atmospheres than the theoretical methods employed by other authors.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, December 28, 1967
185-42-01-02-51

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Appendix A

Symbol List

A	characteristic constant defined in Equation 26
B	characteristic constant defined in Equation 32
$\bar{c}_{p g}$	specific heat at constant pressure
$F_{j w}$	surface function defined in connection with Equation 10
\mathfrak{F}	Froude-number
\bar{g}	gravitational acceleration
\bar{H}_w	depth of water rotating as a solid body; defined in Equation 13
\bar{H}_{max}	maximum altitude of atmosphere
\bar{M}	mass of celestial body
\bar{p}	pressure
\bar{p}_{00}	atmospheric pressure at the free surface of water
\bar{r}	distance from center of rotation
\bar{r}_0	reference length
\bar{R}_g	gas constant
\bar{t}	time
\bar{T}	absolute temperature
$\bar{u}, \bar{v}, \bar{w}$	velocity components parallel to the coordinate axes
$\bar{x}, \bar{y}, \bar{z}$	Cartesian coordinates
$\bar{\Gamma}$	universal gravitational constant
ϵ	small dimensionless length, defined in connection with Equation 34
$\bar{\eta}_w$	disturbance amplitude of water depth
κ	polytropic exponent of a gas

$\bar{\lambda}_w$ wavelength corresponding to $\bar{\eta}_w$
 μ mass ratio
 $\bar{\rho}$ density
 Ψ nondimensional gravitational potential
 $\bar{\Psi}$ gravitational potential
 $\bar{\omega}$ constant angular velocity

Subscripts

i	gas and water	0	reference quantities or quantities taken at the surface of the earth
g	gas	1	bottom of container of water
w	water	2	free surface of water
α, β	first and second celestial body, respectively	II	free surface of water in case of rotation of water as a solid body

All dimensional quantities are marked by a bar above the letters.

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